

HILL'S POTENTIALS IN HÖRMANDER SPACES AND THEIR SPECTRAL GAPS¹

VLADIMIR MIKHAILOTS, VOLODYMYR MOLYBOGA

ABSTRACT. We study the behaviour of the lengths of spectral gaps $\{\gamma_q(n)\}_{n \in \mathbb{N}}$ in a continuous spectrum of the Hill-Schrödinger operators

$$S(q)u = -u'' + q(x)u, \quad x \in \mathbb{R},$$

with 1-periodic real-valued distribution potentials

$$q(x) = \sum_{k \in \mathbb{Z}} \hat{q}(k) e^{ik2\pi x} \in H^{-1}(\mathbb{T}), \quad \hat{q}(k) = \overline{\hat{q}(-k)}, \quad k \in \mathbb{Z},$$

in dependence on the weight ω of the Hörmander space $H^\omega(\mathbb{T}) \ni q$.

Let $h^\omega(\mathbb{N})$ be a Hilbert space of weighted sequences. We prove that

$$\{\hat{q}(\cdot)\} \in h^\omega(\mathbb{N}) \Leftrightarrow \{\gamma_q(\cdot)\} \in h^\omega(\mathbb{N})$$

if a positive, in general non-monotonic, weight $\omega = \{\omega(k)\}_{k \in \mathbb{N}}$ satisfies only two following conditions:

- i) $-1 < \mu(\omega) = \liminf_{k \rightarrow \infty} \frac{\log \omega(k)}{\log k} \leq \rho(\omega) = \limsup_{k \rightarrow \infty} \frac{\log \omega(k)}{\log k} < \infty$,
- ii) $\rho(\omega) < \begin{cases} 1 + 2\mu(\omega) & \text{if } \mu(\omega) \in (-1, 0], \\ 1 + \mu(\omega) & \text{if } \mu(\omega) \in [0, \infty). \end{cases}$

In the case $q \in L^2(\mathbb{T})$, $\omega(k) = (1 + 2k)^s$, $s \in \mathbb{Z}_+$, this result is due to Marchenko and Ostrovskii (1975).

1. INTRODUCTION

Let consider on the complex Hilbert space $L^2(\mathbb{R})$ the Hill-Schrödinger operators

$$(1) \quad S(q)u := -u'' + q(x)u, \quad x \in \mathbb{R},$$

with 1-periodic real-valued potentials

$$q(x) = \sum_{k \in \mathbb{Z}} \hat{q}(k) e^{ik2\pi x} \in L^2(\mathbb{T}, \mathbb{R}), \quad \mathbb{T} := \mathbb{R}/\mathbb{Z}.$$

This means that

$$\sum_{k \in \mathbb{Z}} |\hat{q}(k)|^2 < \infty \quad \text{and} \quad \hat{q}(k) = \overline{\hat{q}(-k)}, \quad k \in \mathbb{Z}.$$

It is well known that the operators $S(q)$ are lower semibounded and self-adjoint. Their spectra are absolutely continuous and have a zone structure [32].

Spectra of the operators $S(q)$, $q \in L^2(\mathbb{T}, \mathbb{R})$, are completely defined by the location of the endpoints of spectral gaps $\{\lambda_0(q), \lambda_n^\pm(q)\}_{n=1}^\infty$, which satisfy the inequalities:

$$(2) \quad -\infty < \lambda_0(q) < \lambda_1^-(q) \leq \lambda_1^+(q) < \lambda_2^-(q) \leq \lambda_2^+(q) < \dots$$

¹The investigation is partially supported by DFFD of Ukraine under grant $\Phi 28.1/017$.

2000 *Mathematics Subject Classification*. Primary 34L40; Secondary 47A10, 47A75.

Key words and phrases. Hill-Schrödinger operators, singular potentials, spectral gaps, Hörmander spaces.

For even/odd numbers $n \in \mathbb{Z}_+$ the endpoints of spectral gaps $\{\lambda_0(q), \lambda_n^\pm(q)\}_{n=1}^\infty$ are eigenvalues of the periodic/semiperiodic problems on the interval $[0, 1]$:

$$S_\pm(q)u := -u'' + q(x)u = \lambda u,$$

$$\text{Dom}(S_\pm(q)) := \left\{ u \in H^2[0, 1] \mid u^{(j)}(0) = \pm u^{(j)}(1), j = 0, 1 \right\}.$$

Interiors of spectral bands (stability or tied zones)

$$\mathcal{B}_0(q) := (\lambda_0(q), \lambda_1^-(q)), \quad \mathcal{B}_n(q) := (\lambda_n^+(q), \lambda_{n+1}^-(q)), \quad n \in \mathbb{N},$$

together with the *collapsed* gaps

$$\lambda = \lambda_n^+ = \lambda_n^-, \quad n \in \mathbb{N}$$

are characterized as a locus of those real $\lambda \in \mathbb{R}$ for which all solutions of the equation $S(q)u = \lambda u$ are bounded. *Open* spectral gaps (instability or forbidden zones)

$$\mathcal{G}_0(q) := (-\infty, \lambda_0(q)), \quad \mathcal{G}_n(q) := (\lambda_n^-(q), \lambda_n^+(q)) \neq \emptyset, \quad n \in \mathbb{N}$$

are a locus of those real $\lambda \in \mathbb{R}$ for which any nontrivial solution of the equation $S(q)u = \lambda u$ is unbounded.

We are going to characterize the behaviour of the lengths of spectral gaps

$$\gamma_q(n) := \lambda_n^+(q) - \lambda_n^-(q), \quad n \in \mathbb{N}$$

of the Hill-Schrödinger operators $S(q)$ in terms of the behaviour of the Fourier coefficients $\{\hat{q}(n)\}_{n \in \mathbb{N}}$ of the potentials q with respect to appropriate weight spaces, that is by means of potential regularity.

For $L^2(\mathbb{T}, \mathbb{R})$ -potentials fundamental result in this problem follows from the Marchenko and Ostrovskii paper [18] (see also [17]):

$$(3) \quad q \in H^s(\mathbb{T}, \mathbb{R}) \Leftrightarrow \sum_{n \in \mathbb{N}} (1 + 2n)^{2s} \gamma_q^2(n), \quad s \in \mathbb{Z}_+,$$

where $H^s(\mathbb{T}, \mathbb{R})$, $s \in \mathbb{Z}_+$, denotes the Sobolev spaces of 1-periodic real-valued functions on the circle \mathbb{T} .

To characterize regularity of potentials in the finer way we will use the real Hörmander spaces

$$H^\omega(\mathbb{T}, \mathbb{R}) := \left\{ f = \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{ik2\pi x} \mid \hat{f}(k) = \overline{\hat{f}(-k)}, k \in \mathbb{Z}, \sum_{k \in \mathbb{N}} \omega^2(k) |\hat{f}(k)|^2 < \infty \right\},$$

where $\omega(\cdot)$ is a positive weight. In the case of the Sobolev spaces it is a power one. Such definition of the real Hörmander spaces on the circle completely corresponds to the theory of function spaces on a smooth closed manifold [28, 29].

Djakov, Mityagin [6], Pöschel [30, 31] extended the Marchenko-Ostrovskii Theorem (3) to the general class of weights $\Omega = \{\Omega(k)\}_{k \in \mathbb{N}}$ satisfying the following conditions:

- i) $\Omega(k) \nearrow \infty$, $k \in \mathbb{N}$; (monotonicity)
- ii) $\Omega(k + m) \leq \Omega(k)\Omega(m) \quad \forall k, m \in \mathbb{N}$; (submultiplicity)
- iii) $\frac{\log \Omega(k)}{k} \searrow 0$, $k \rightarrow \infty$, (subexponentiality).

For such weights they proved that

$$(4) \quad q \in H^\Omega(\mathbb{T}, \mathbb{R}) \Leftrightarrow \{\gamma_q(\cdot)\} \in h^\Omega(\mathbb{N}).$$

Here $h^\Omega(\mathbb{N})$ is the Hilbert space of weighted sequences generated by the weight $\Omega(\cdot)$.

Earlier Kappeler, Mityagin [13] proved the direct implication in (4) under the only assumption of submultiplicity. In the special cases of the Abel-Sobolev weights, the Gevrey weights and the slowly increasing weights the relationship (4) was established by Kappeler, Mityagin

[12] (\Rightarrow) and Djakov, Mityagin [4, 5] (\Leftarrow). Detailed exposition of these results is given in the survey [6]. It should be noted that Pöschel [30, 31], Kappeler, Mityagin [12, 13] and Djakov, Mityagin [5, 6] studied also the more general case of complex-valued potentials.

For the power weights

$$w_s = \{w_s(k)\}_{k \in \mathbb{N}} : \quad w_s(k) := (1 + 2k)^s, \quad s \in \mathbb{R}$$

it is convenient to use shorter notation

$$H^{\omega_s}(\mathbb{T}) \equiv H^s(\mathbb{T}), \quad h^{\omega_s}(\mathbb{N}) \equiv h^s(\mathbb{N}).$$

After the celebrated Kronig and Penney paper [16] the Schrödinger operators with (periodic) distributions as potentials came into mathematical physics. The following development of quantum mechanics stimulated active growth of this branch of science (see the bibliography of the monographs [1, 2], which counts several hundreds of physical and mathematical works).

In this paper we study the Hill-Schrödinger operators $S(q)$ with 1-periodic real-valued distribution potentials q from the negative Sobolev space $H^{-1}(\mathbb{T}, \mathbb{R})$:

$$(5) \quad q(x) = \sum_{k \in \mathbb{Z}} \hat{q}(k) e^{ik2\pi x} \in H^{-1}(\mathbb{T}, \mathbb{R}).$$

This means that

$$\sum_{k \in \mathbb{N}} (1 + 2k)^{2s} |\hat{q}(k)|^2 < \infty, \quad s = -1, \quad \text{and} \quad \hat{q}(k) = \overline{\hat{q}(-k)}, \quad k \in \mathbb{Z}. \quad (6_s)$$

All pseudo-functions, measures, pseudo-measures and some even more singular distributions on the circle satisfy this condition.

Under the assumption (5) the operators (1) can be well defined on the complex Hilbert space $L^2(\mathbb{R})$ in the following basic ways:

- as form-sum operators;
- as quasi-differential operators (minimal operators, maximal operators, the Friedrichs extensions of the minimal operators);
- as limits of operators with smooth 1-periodic potentials in the norm resolvent sense.

Equivalence of all these definitions was proved in the paper [22].

The Hill-Schrödinger operators $S(q)$, $q \in H^{-1}(\mathbb{T}, \mathbb{R})$, are lower semibounded and self-adjoint, their spectra are absolutely continuous and have a band and gap structure as in the classical case of $L^2(\mathbb{T}, \mathbb{R})$ -potentials, [11, 15, 7, 22, 24, 10]. The endpoints of spectral gaps $\{\lambda_0(q), \lambda_n^\pm(q)\}_{n=1}^\infty$ satisfy the inequalities (2). For even/odd numbers $n \in \mathbb{Z}_+$ they are eigenvalues of the periodic/semiperiodic problems on the interval $[0, 1]$ [22, Theorem C].

In the paper [23] we extended the Marchenko-Ostrovskii Theorem (3) to the case of singular potentials $q \in H^{-1+}(\mathbb{T}, \mathbb{R})$. This means that q satisfies (6_s) with some $s > -1$. We proved that

$$(7) \quad q \in H^s(\mathbb{T}, \mathbb{R}) \Leftrightarrow \{\gamma_q(\cdot)\} \in h^s(\mathbb{N}), \quad s \in (-1, \infty).$$

The case $s \in (-1, 0]$ was earlier treated in [21, 26].

Djakov, Mityagin [8, 9] extended the latter statement to the limiting case $s = -1$:

$$(8) \quad q \in H^s(\mathbb{T}, \mathbb{R}) \Leftrightarrow \{\gamma_q(\cdot)\} \in h^s(\mathbb{N}), \quad s \in [-1, \infty),$$

under the *a priori* assumption $q \in H^{-1}(\mathbb{T}, \mathbb{R})$. Moreover, they extended the result (4) to the case of potentials $q \in H^{-1}(\mathbb{T}, \mathbb{R})$ and more general weights:

$$(9) \quad q \in H^{\Omega^*}(\mathbb{T}, \mathbb{R}) \Leftrightarrow \{\gamma_q(\cdot)\} \in h^{\Omega^*}(\mathbb{N}), \quad \Omega^* := \left\{ \frac{\Omega(k)}{1 + 2k} \right\}_{k \in \mathbb{N}},$$

where the weights $\Omega = \{\Omega(k)\}_{k \in \mathbb{N}}$ are supposed to be monotonic, submultiplicative and subexponential.

2. MAIN RESULTS

The aim of this paper is to extend the result (8) to a more extensive class of weights, for which the conditions of regularity of the weight behaviour may not hold, and to supplement the result (9).

For convenience of formulation of the results we introduce the following definition.

Definition. Let the set $X \subset H^{-1}(\mathbb{T}, \mathbb{R})$. We write $\omega \in \text{MO}(X)$ if

$$q \in H^\omega(\mathbb{T}, \mathbb{R}) \Leftrightarrow \{\gamma_q(\cdot)\} \in h^\omega(\mathbb{N}) \quad \forall q \in X.$$

It is easy to see that

$$X \supset Y \Rightarrow \text{MO}(X) \subset \text{MO}(Y),$$

and

$$(3) \Leftrightarrow \omega_s \in \text{MO}(L^2(\mathbb{T})), \quad s \in \mathbb{Z}_+,$$

$$(4) \Leftrightarrow \Omega \in \text{MO}(L^2(\mathbb{T})),$$

$$(8) \Leftrightarrow \omega_s \in \text{MO}(H^{-1}(\mathbb{T})), \quad s \in [-1, \infty),$$

$$(9) \Leftrightarrow \Omega^* \in \text{MO}(H^{-1}(\mathbb{T})).$$

Further, let us recall that *lower order* $\mu(\omega)$ and *upper order* $\rho(\omega)$ of a weight sequence $\omega = \{\omega(k)\}_{k \in \mathbb{N}}$ are defined as follows:

$$\mu \equiv \mu(\omega) := \liminf_{k \rightarrow \infty} \frac{\log \omega(k)}{\log k}, \quad \rho \equiv \rho(\omega) := \limsup_{k \rightarrow \infty} \frac{\log \omega(k)}{\log k}.$$

In what follows we will consider only weights satisfying the conditions:

$$-1 < \mu(\omega) \leq \rho(\omega) < \infty.$$

Such weights have not more than a power growth. The following statement is the main result of this paper.

Theorem 1. *Let $q \in H^{-1}(\mathbb{T}, \mathbb{R})$ and the weight $\omega = \{\omega(k)\}_{k \in \mathbb{N}}$ satisfy conditions:*

$$(10) \quad -1 < \mu(\omega) \leq \rho(\omega) < \infty,$$

$$(11) \quad \rho(\omega) < \begin{cases} 1 + 2\mu(\omega) & \text{if } \mu(\omega) \in (-1, 0], \\ 1 + \mu(\omega) & \text{if } \mu(\omega) \in [0, \infty). \end{cases}$$

Then $\omega \in \text{MO}(H^{-1}(\mathbb{T}))$.

Corollary 1.1. *Let for the weight $\omega = \{\omega(k)\}_{k \in \mathbb{N}}$ exist the order*

$$\lim_{k \rightarrow \infty} \frac{\log \omega(k)}{\log k} = s \in (-1, \infty).$$

Then $\omega \in \text{MO}(H^{-1}(\mathbb{T}))$.

From Corollary 1.1 we receive the following result.

Corollary 1.2 (cf. [23]). *Let the weight $\omega = \{\omega(k)\}_{k \in \mathbb{N}}$ be a regular varying sequence on $+\infty$ in the Karamata sense with the index $s \in (-1, \infty)$. Then $\omega \in \text{MO}(H^{-1}(\mathbb{T}))$.*

Note that the assumption of Corollary 1.2 holds, for instance, for the weight

$$\omega(k) = (1 + 2k)^s (\log(1 + k))^{r_1} (\log \log(1 + k))^{r_2} \dots (\log \log \dots \log(1 + k))^{r_p},$$

$$s \in (-1, \infty), \quad \{r_1, \dots, r_p\} \subset \mathbb{R}, \quad p \in \mathbb{N},$$

see the monographs [3, 33].

Theorem 1 extends the statement (8) and Corollary 1.2 to the case of non-regularly varying weights.

The following Example A shows that statement (9) does not cover Corollary 1.1 and all the more Theorem 1.

Example A. Let $s \in (-1, \infty)$. Set

$$w(k) := \begin{cases} k^s \log(1+k) & \text{if } k \in 2\mathbb{N}, \\ k^s & \text{if } k \in (2\mathbb{N}-1). \end{cases}$$

Then the weight $\omega = \{\omega(k)\}_{k \in \mathbb{N}}$ satisfies the conditions of Corollary 1.1. But one can prove that the weight

$$\omega^* := \{(1+2k)\omega(k)\}_{k \in \mathbb{N}}$$

is not equivalent to any monotonic weight.

3. PRELIMINARIES

Here, for convenience, we define Hilbert spaces of weighted two-sided sequences and formulate the Convolution Lemma 2.

For every positive sequence $\omega = \{\omega(k)\}_{k \in \mathbb{N}}$ there exists its unique extension on \mathbb{Z} which is a two-sided sequence satisfying the conditions:

- i) $\omega(0) = 1$;
- ii) $\omega(-k) = \omega(k) \quad \forall k \in \mathbb{N}$;
- iii) $\omega(k) > 0 \quad \forall k \in \mathbb{Z}$.

Let

$$h^\omega(\mathbb{Z}) \equiv h^\omega(\mathbb{Z}, \mathbb{C})$$

be the Hilbert space of two-sided sequences:

$$\begin{aligned} h^\omega(\mathbb{Z}) &:= \left\{ a = \{a(k)\}_{k \in \mathbb{Z}} \left| \sum_{k \in \mathbb{Z}} \omega^2(k) |a(k)|^2 < \infty \right. \right\}, \\ (a, b)_{h^\omega(\mathbb{Z})} &:= \sum_{k \in \mathbb{Z}} \omega^2(k) a(k) \overline{b(k)}, \quad a, b \in h^\omega(\mathbb{Z}), \\ \|a\|_{h^\omega(\mathbb{Z})} &:= (a, a)_{h^\omega(\mathbb{Z})}^{1/2}, \quad a \in h^\omega(\mathbb{Z}). \end{aligned}$$

Basic weights which we use are the power ones:

$$w_s = \{w_s(k)\}_{k \in \mathbb{Z}} : \quad w_s(k) = (1 + 2|k|)^s, \quad s \in \mathbb{R}.$$

In this case it is convenient to use shorter notations:

$$h^{\omega_s}(\mathbb{Z}) \equiv h^s(\mathbb{Z}), \quad s \in \mathbb{R}.$$

Operation of convolution for two-sided sequences

$$a = \{a(k)\}_{k \in \mathbb{Z}} \quad \text{and} \quad b = \{b(k)\}_{k \in \mathbb{Z}}$$

is formally defined as follows:

$$\begin{aligned} (a, b) &\mapsto a * b, \\ (a * b)(k) &:= \sum_{j \in \mathbb{Z}} a(k-j) b(j), \quad k \in \mathbb{Z}. \end{aligned}$$

Sufficient conditions for the convolution to exist as a continuous map are given by the following known lemma, see for example [14, 27].

Lemma 2 (The Convolution Lemma). *Let $s, r \geq 0$, and $t \leq \min(s, r)$, $t \in \mathbb{R}$. If $s + r - t > 1/2$ then the convolution $(a, b) \mapsto a * b$ is well defined as a continuous map acting in the spaces:*

- (a) $h^s(\mathbb{Z}) \times h^r(\mathbb{Z}) \rightarrow h^t(\mathbb{Z})$,
- (b) $h^{-t}(\mathbb{Z}) \times h^s(\mathbb{Z}) \rightarrow h^{-r}(\mathbb{Z})$.

In the case $s + r - t < 1/2$ this statement fails to hold.

4. THE PROOFS

Basic point of our proof of Theorem 1 is sharp asymptotic formulae for the lengths of spectral gaps $\{\gamma_q(n)\}_{n \in \mathbb{N}}$ of the Hill-Schrödinger operators $S(q)$.

Lemma 3. *The lengths of spectral gaps $\{\gamma_q(n)\}_{n \in \mathbb{N}}$ of the Hill-Schrödinger operators $S(q)$ with $q \in H^s(\mathbb{T}, \mathbb{R})$, $s \in (-1, \infty)$, uniformly on the bounded sets of potentials q in the corresponding Sobolev spaces $H^s(\mathbb{T})$ for $n \geq n_0$, $n_0 = n_0(\|q\|_{H^s(\mathbb{T})})$, satisfy the following asymptotic formulae:*

$$(12) \quad \gamma_q(n) = 2|\widehat{q}(n)| + h^{1+2s-\varepsilon}(n), \quad \varepsilon > 0 \quad \text{if } s \in (-1, 0],$$

$$(13) \quad \gamma_q(n) = 2|\widehat{q}(n)| + h^{1+s}(n) \quad \text{if } s \in [0, \infty).$$

Proof of Lemma 3. The asymptotic estimates (12) were established by the authors in the paper [23, Theorem 1] (see also [21, 26]) by method of the isospectral transformation of the problem [25, 19, 20, 21, 26].

The asymptotic formulae (13) follow from [13, Theorem 1.2] due to the Convolution Lemma 2 (see also [13, Appendix]). Indeed, from [13, Theorem 1.2] with $q \in H^s(\mathbb{T}, \mathbb{R})$, $s \in [0, \infty)$, we get

$$(14) \quad \sum_{n \in \mathbb{N}} (1 + 2n)^{2(1+s)} \left(\min_{\pm} \left| \gamma_q(n) \pm 2\sqrt{(\widehat{q} + \varrho)(-n)(\widehat{q} + \varrho)(n)} \right| \right)^2 \leq C(\|q\|_{H^s(\mathbb{T})}),$$

where

$$\varrho(n) := \frac{1}{\pi^2} \sum_{j \in \mathbb{Z} \setminus \{\pm n\}} \frac{\widehat{q}(n-j)\widehat{q}(n+j)}{(n-j)(n+j)}.$$

Without losing generality we assume that

$$(15) \quad \widehat{q}(0) := 0.$$

Taking into account that the potentials q are real-valued we have

$$\widehat{q}(k) = \overline{\widehat{q}(-k)}, \quad \varrho(k) = \overline{\varrho(-k)}, \quad k \in \mathbb{Z}.$$

Then from (14) we get the estimates

$$(16) \quad \{\gamma_q(n) - 2|\widehat{q}(n) + \varrho(n)|\}_{n \in \mathbb{N}} \in h^{1+s}(\mathbb{N}).$$

Further, as by assumption $q \in H^s(\mathbb{T}, \mathbb{R})$, that is $\{\widehat{q}(k)\}_{k \in \mathbb{Z}} \in h^s(\mathbb{Z})$, then

$$\left\{ \frac{\widehat{q}(k)}{k} \right\}_{k \in \mathbb{Z}} \in h^{1+s}(\mathbb{Z}), \quad s \in [0, \infty),$$

taking into account (15). Applying the Convolution Lemma 2 we obtain

$$(17) \quad \begin{aligned} \varrho(n) &= \frac{1}{\pi^2} \sum_{j \in \mathbb{Z}} \frac{\widehat{q}(n-j)\widehat{q}(n+j)}{(n-j)(n+j)} = \frac{1}{\pi^2} \sum_{j \in \mathbb{Z}} \frac{\widehat{q}(2n-j)}{2n-j} \cdot \frac{\widehat{q}(j)}{j} \\ &= \left(\left\{ \frac{\widehat{q}(k)}{k} \right\}_{k \in \mathbb{Z}} * \left\{ \frac{\widehat{q}(k)}{k} \right\}_{k \in \mathbb{Z}} \right) (2n) \in h^{1+s}(\mathbb{N}). \end{aligned}$$

Finally, from (16) and (17) we get the necessary estimates (13).

The proof of Lemma 3 is complete. \square

Proof of Theorem 1. Let $q \in H^{-1}(\mathbb{T}, \mathbb{R})$ and $\omega = \{\omega(k)\}_{k \in \mathbb{N}}$ be a given weight satisfying the conditions (10) and (11) of Theorem 1. We need to prove the statement

$$(18) \quad q \in H^\omega(\mathbb{T}, \mathbb{R}) \Leftrightarrow \{\gamma_q(\cdot)\} \in h^\omega(\mathbb{N}).$$

From formula (10) and definition of the lower and upper orders of a weight sequence we conclude that for the given weight $\omega = \{\omega(k)\}_{k \in \mathbb{N}}$ the following estimates are fulfilled:

$$k^{\mu-\delta} \ll \omega(k) \ll k^{\rho+\delta}, \quad -1 < \mu \leq \rho < \infty, \delta > 0.$$

Hence, the continuous embeddings

$$(19) \quad H^{\rho+\delta}(\mathbb{T}) \hookrightarrow H^\omega(\mathbb{T}) \hookrightarrow H^{\mu-\delta}(\mathbb{T}),$$

$$(20) \quad h^{\rho+\delta}(\mathbb{N}) \hookrightarrow h^\omega(\mathbb{N}) \hookrightarrow h^{\mu-\delta}(\mathbb{N}), \quad -1 < \mu \leq \rho < \infty, \delta > 0$$

are valid because of

$$(21) \quad H^{\omega_1}(\mathbb{T}) \hookrightarrow H^{\omega_2}(\mathbb{T}), \quad h^{\omega_1}(\mathbb{N}) \hookrightarrow h^{\omega_2}(\mathbb{N}) \quad \text{if only} \quad \omega_1 \gg \omega_2.$$

Let $q \in H^\omega(\mathbb{T}, \mathbb{R})$, then from (19) we obtain that $q \in H^{\mu-\delta}(\mathbb{T}, \mathbb{R})$, $\delta > 0$. Due to arbitrary choice of $\delta > 0$ we may choose it so that:

$$\begin{aligned} \mu - \delta &> 0 \quad \text{if} \quad \mu > 0, \\ \mu - \delta &> -1 \quad \text{otherwise.} \end{aligned}$$

Further, using Lemma 3 we get the following asymptotic formulae for the lengths of spectral gaps:

$$(22) \quad \gamma_q(n) = 2|\widehat{q}(n)| + h^{1+2(\mu-\delta)-\varepsilon}(n), \quad \varepsilon > 0 \quad \text{if} \quad \mu - \delta \in (-1, 0],$$

$$(23) \quad \gamma_q(n) = 2|\widehat{q}(n)| + h^{1+\mu-\delta}(n) \quad \text{if} \quad \mu - \delta \in [0, \infty).$$

Now, due to possibility of arbitrary choice of $\delta > 0$ and $\varepsilon > 0$ we may choose them so that:

$$(24) \quad 1 + 2(\mu - \delta) - \varepsilon > \rho + \delta \quad \text{if} \quad \mu - \delta \in (-1, 0],$$

$$(25) \quad 1 + \mu - \delta > \rho + \delta \quad \text{if} \quad \mu - \delta \in [0, \infty).$$

This choice is possible as (10) and (11) hold.

Taking into account (24), (25) and using (21), from (22), (23) we get estimates

$$\gamma_q(n) = 2|\widehat{q}(n)| + h^{\rho+\delta}(n).$$

From the latter formula and formula (20) we finally derive

$$(26) \quad \gamma_q(n) = 2|\widehat{q}(n)| + h^\omega(n).$$

Hence, as by assumption $q \in H^\omega(\mathbb{T}, \mathbb{R})$, and as consequence $\{\widehat{q}(\cdot)\} \in h^\omega(\mathbb{N})$, we get $\{\gamma_q(\cdot)\} \in h^\omega(\mathbb{N})$.

The implication (\Rightarrow) in (18) has been proved.

Conversely, let $\{\gamma_q(\cdot)\} \in h^\omega(\mathbb{N})$. Then applying (20) we have $\{\gamma_q(\cdot)\} \in h^{\mu-\delta}(\mathbb{N})$, $\delta > 0$:

$$\begin{aligned} \mu - \delta &> 0 \quad \text{if} \quad \mu > 0, \\ \mu - \delta &> -1 \quad \text{otherwise.} \end{aligned}$$

Now, applying (8) we conclude that $q \in H^{\mu-\delta}(\mathbb{T}, \mathbb{R})$.

We have already proved the implication

$$q \in H^{\mu-\delta}(\mathbb{T}, \mathbb{R}) \Rightarrow (26).$$

So we have

$$\gamma_q(n) = 2|\widehat{q}(n)| + h^\omega(n),$$

and hence $\{\widehat{q}(\cdot)\} \in h^\omega(\mathbb{N})$, i.e., $q \in H^\omega(\mathbb{T}, \mathbb{R})$.

The implication (\Leftarrow) in (18) has been proved.

The proof of Theorem 1 is complete.

ACKNOWLEDGMENTS

The authors would like to express their gratitude to Professors V.A. Marchenko and F.S. Rofo-Beketov, for their attention to this paper.

REFERENCES

- [1] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden, *Solvable Models in Quantum Mechanics*, Springer-Verlag, New York, 1988.
- [2] S. Albeverio, P. Kurasov, *Singular Perturbations of Differential Operators: Solvable Schrödinger Type Operators*, University Press, Cambridge, 1999.
- [3] N. Bingham, C. Goldie, J. Teugels, *Regular Variation*, Encyclopedia of Math. and its Appl., vol. 27, Cambridge University Press, Cambridge, etc., 1989.
- [4] P. Djakov, B. Mityagin, *Smoothness of Schrödinger operator potential in the case of Gevrey type asymptotics of the gaps*, J. Funct. Anal. **195** (2002), 89–128.
- [5] P. Djakov, B. Mityagin, *Spectral triangles of Schrödinger operators with complex potentials*, Selecta Math. (N.S.) **9** (2003), no. 4, 495–528.
- [6] P. Djakov, B. Mityagin, *Instability zones of one-dimensional periodic Schrödinger and Dirac operators*, Uspekhi Mat. Nauk **61** (2006), no. 4, 77–182. (Russian); English Transl. in Russian Math. Surveys **61** (2006), no. 4, 663–766.
- [7] P. Djakov, B. Mityagin, *Fourier method for one dimensional Schrödinger operators with singular periodic potentials*, arXiv: math.SP/0710.0237 1 Oct 2007, 1–39.
- [8] P. Djakov, B. Mityagin, *Spectral gap asymptotics of one-dimensional Schrödinger operators with singular periodic potentials*, Integral Transforms and Special Functions **20** (2009), no. 3-4, 265–273.
- [9] P. Djakov, B. Mityagin, *Spectral gaps of Schrödinger operators with periodic singular potentials*, arXiv: math.SP/0903.5210 30 March 2009, 1–78.
- [10] F. Gesztesy, W. Kirsch, *One-dimensional Schrödinger operators with interactions singular on a discrete set*, J. Reine Angew. Math. **362** (1985), 28–50.
- [11] R. Hryniv, Ya. Mykytyuk, *1-D Schrödinger operators with periodic singular potentials*, Methods Funct. Anal. Topology **7** (2001), no. 4, 31–42.
- [12] T. Kappeler, B. Mityagin, *Gap estimates of the spectrum of Hill's equation and actions variables for KdV* , Trans. AMS **351** (1999), no. 2, 619–646.
- [13] T. Kappeler, B. Mityagin, *Estimates for periodic and Dirichlet eigenvalues of the Schrödinger operator*, SIAM J. Math. Anal. **33** (2001), no. 1, 113–152.
- [14] T. Kappeler, C. Möhr, *Estimates for periodic and Dirichlet eigenvalues of the Schrödinger operator with singular potentials*, J. Funct. Anal. **186** (2001), 62–91.
- [15] E. Korotyaev, *Characterization of the spectrum of Schrödinger operators with periodic distributions*, Int. Math. Res. Not. **37** (2003), 2019–2031.
- [16] R. de L. Kronig, W. Penney, *Quantum mechanics of electrons in crystal lattices*, Proc. Royal Society (London) A **130** (1931), 499–513.
- [17] V. Marchenko, *Sturm-Liouville Operators and Applications*, Birkhäuser Verlag, Basel, 1986. (Russian edition: Naukova Dumka, Kiev, 1977)
- [18] V. Marchenko, I. Ostrovskii, *A characterization of the spectrum of Hill's operator*, Matem. Sbornik **97** (1975), no. 4, 540–606. (Russian); English Transl. in Math. USSR-Sb. **26** (1975), no. 4, 493–554.
- [19] V. Mikhailets, V. Molyboga, *Singular eigenvalue problems on the circle*, Methods Funct. Anal. Topology **10** (2004), no. 3, 44–53.
- [20] V. Mikhailets, V. Molyboga, *Uniform estimates for the semi-periodic eigenvalues of the singular differential operators*, Methods Funct. Anal. Topology **10** (2004), no. 4, 30–57.
- [21] V. Mikhailets, V. Molyboga, *The spectral problems over the periodic classes of distributions*, Preprint, Institute of Mathematics of NAS of Ukraine, Kyiv, 2004, 46 p. (Ukrainian)
- [22] V. Mikhailets, V. Molyboga, *One-dimensional Schrödinger operators with singular periodic potentials*, Methods Funct. Anal. Topology **14** (2008), no. 2, 184–200.
- [23] V. Mikhailets, V. Molyboga, *Spectral gaps of the one-dimensional Schrödinger operators with singular periodic potentials*, Methods Funct. Anal. Topology **15** (2009), no. 1, 31–40.
- [24] V. Mikhailets, A. Sobolev, *Common eigenvalue problem and periodic Schrödinger operators*, J. Funct. Anal. **165** (1999), no. 1, 150–172.
- [25] V. Molyboga, *Estimates for periodic eigenvalues of the differential operator $(-1)^m d^{2m}/dx^{2m} + V$ with V - distribution*, Methods Funct. Anal. Topology **9** (2003), no. 2, 163–178.
- [26] V. Molyboga, *Spectra of periodic problems with distributions*, PhD Thesis, Kyiv, 2005, 167 p. (Ukrainian)
- [27] C. Möhr, *Schrödinger Operators with Singular Potentials on the Circle: Spectral Analysis and Applications*, Thesis at the University of Zürich, 2001, 134 p.

- [28] V. Mikhailets, A. Murach, *Interpolation with a function parameter and refined scale of spaces*, Methods Funct. Anal. Topology **14** (2008), no. 1, 81–100.
- [29] V. Mikhailets, A. Murach, *On the elliptic operators on a closed compact manifold*, Reports of NAS of Ukraine (2009), no. 3, 29–35. (Russian)
- [30] J. Pöschel, *Hill's potentials in weighted Sobolev spaces and their spectral gaps*, Preprint, Stuttgarter Mathematische Berichte, Stuttgart, 2004, 28 p.
- [31] J. Pöschel, *Hill's potentials in weighted Sobolev spaces and their spectral gaps*, In: W. Craig (ed), Hamiltonian Systems and applications, Springer, 2008, 421–430.
- [32] M. Reed, B. Simon, Methods of Modern Mathematical Physics: Vols 1-4, Academic Press, New York, etc., 1972–1978, V. 4: Analysis of Operators, 1972. (Russian edition: Mir, Moscow, 1982)
- [33] E. Seneta, Regularly Varying Functions, Lecture Notes in Math., vol. 508, Springer-Verlag, Berlin, 1976. (Russian edition: Nauka, Moscow, 1985)

INSTITUTE OF MATHEMATICS OF NAS OF UKRAINE, TERESHCHENKIVSKA STR., 3, KYIV-4, UKRAINE, 01601

E-mail address, Vladimir Mikhailets: `mikhailets@imath.kiev.ua`

E-mail address, Volodymyr Molyboga: `molyboga@imath.kiev.ua`